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Exact evaluation of a path integral relating to an electron gas in a random potential

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Abstract. The path integral of an oscillator with memory occurring in the theory of an electron gas in a random potential is evaluated exactly. The analysis simply reduces the problem to that of averaging the propagator of a forced harmonic oscillator using a gaussian probability for the external force.

1. Introduction

Bezák (1970) in connection with a path integral theory of an electron gas in a random potential was faced with the evaluation of the following path integral:

$$G(\boldsymbol{x}, \beta|\boldsymbol{x}_0, 0) = \int_{\boldsymbol{x}(0)=\boldsymbol{x}_0}^{\boldsymbol{x}(\beta\hbar)=\boldsymbol{x}} \mathscr{D}[\boldsymbol{x}(u)] \exp\left(-\frac{1}{\hbar}S[\boldsymbol{x}(u)]\right) \tag{1.1}$$

where $\mathscr{D}[\boldsymbol{x}(u)]$ is the usual Feynman path differential measure (when working with imaginary time $\beta\hbar/\mathrm{i}$; $\beta = 1/kT$) and $S[\boldsymbol{x}(u)]$ is the action for an oscillator with memory given by:

$$S[\boldsymbol{x}(u)] = \int_0^{\beta\hbar} \frac{m}{2}\dot{\boldsymbol{x}}^2(u)\,\mathrm{d}u + \frac{m}{4\beta\hbar}\Omega^2 \int_0^{\beta\hbar} \mathrm{d}u \int_0^{\beta\hbar} \mathrm{d}u'(\boldsymbol{x}(u)-\boldsymbol{x}(u'))^2. \tag{1.2}$$

As is well known in the case of quadratic action functionals, as at present, the propagator takes the form:

$$G(\boldsymbol{x}, \beta|\boldsymbol{x}_0, 0) = \Phi(\beta)\exp\left(-\frac{1}{\hbar}S_{\mathrm{c}}(\boldsymbol{x}, \beta|\boldsymbol{x}_0, 0)\right) \tag{1.3}$$

where S_{c} is the action of an electron along the classical path from $\boldsymbol{x}(0) = \boldsymbol{x}_0$ to $\boldsymbol{x}(\beta\hbar) = \boldsymbol{x}$, and $\Phi(\beta)$ is a function independent of the spatial coordinates but fully determined by the parameter $\beta = 1/kT$. Bezák (1970) obtained exactly the exponential expression of (1.3) but approximated the pre-exponential factor $\Phi(\beta)$ following a rather complicated way. In a follow-up paper (Bezák 1971), entirely devoted to the evaluation of $\Phi(\beta)$, he gave an improved version of $\Phi(\beta)$ in terms of an infinite product of factors, implicitly defined through the roots of a transcendental equation. This he further approximated by a simpler expression but for a certain range of the dimensionless parameter $\beta\hbar\Omega$.

The purpose of our paper is to present a direct and physical method which leads to a simple and exact result for the propagator (1.1) valid for the whole range of the parameter $\beta\hbar\Omega$.

2. The evaluation

As a first step towards our evaluation we express the action (1.2) as:

$$S[\boldsymbol{x}(u)] = \int_0^{\beta\hbar} \tfrac{1}{2}m(\dot{\boldsymbol{x}}^2(u)+\Omega^2\boldsymbol{x}^2(u))\,\mathrm{d}u - \left(\frac{m\Omega^2}{2\beta\hbar}\right)\left(\int_0^{\beta\hbar} \boldsymbol{x}(u)\,\mathrm{d}u\right)^2. \tag{2.1}$$

Inserting (2.1) into (1.1) our path integral takes the form:

$$G(\boldsymbol{x},\beta|\boldsymbol{x}_0,0) = \int_{\boldsymbol{x}(0)=\boldsymbol{x}_0}^{\boldsymbol{x}(\beta\hbar)=\boldsymbol{x}} \mathscr{D}[\boldsymbol{x}(u)]\exp\left(-\frac{1}{\hbar}\int_0^{\beta\hbar}\tfrac{1}{2}m(\dot{\boldsymbol{x}}^2(u)+\Omega^2\boldsymbol{x}^2(u))\,\mathrm{d}u\right)$$
$$\times\exp\left[\frac{1}{\hbar}\left(\frac{m\Omega^2}{2\beta\hbar}\right)\left(\int_0^{\beta\hbar}\boldsymbol{x}(u)\,\mathrm{d}u\right)^2\right]. \tag{2.2}$$

The awkward part of the path integral (2.2) is the last exponential functional which involves off-diagonal terms in full. However, the difficulty can be overcome as follows: we generate this functional through averaging a linear exponential functional, involving an auxiliary random force $\boldsymbol{f}$ independent of $\beta\hbar$, using an appropriate gaussian distribution for the random force.

More explicitly we have:

$$\exp\left[\frac{1}{\hbar}\left(\frac{m\Omega^2}{2\beta\hbar}\right)\left(\int_0^{\beta\hbar}\boldsymbol{x}(u)\,\mathrm{d}u\right)^2\right]$$
$$= \int \mathrm{d}\boldsymbol{f}\exp\left(-\frac{1}{\hbar}\int_0^{\beta\hbar}\boldsymbol{f}\,.\,\boldsymbol{x}(u)\,\mathrm{d}u\right)\left(\frac{2\pi m\Omega^2}{\beta}\right)^{-3/2}\exp\left(-\frac{\beta}{2m\Omega^2}\boldsymbol{f}^2\right). \tag{2.3}$$

Next we insert (2.3) into (2.2) and obtain:

$$G(\boldsymbol{x},\beta|\boldsymbol{x}_0,0) = \int \mathrm{d}\boldsymbol{f}\left(\frac{2\pi m\Omega^2}{\beta}\right)^{-3/2}\exp\left(-\frac{\beta}{2m\Omega^2}\boldsymbol{f}^2\right)G_0(\boldsymbol{x},\beta|\boldsymbol{x}_0,0;\boldsymbol{f}) \tag{2.4}$$

where

$$G_0(\boldsymbol{x},\beta|\boldsymbol{x}_0,0;\boldsymbol{f})$$
$$= \int_{\boldsymbol{x}(0)=\boldsymbol{x}_0}^{\boldsymbol{x}(\beta\hbar)=\boldsymbol{x}}\mathscr{D}[\boldsymbol{x}(u)]\exp\left(-\frac{1}{\hbar}\int_0^{\beta\hbar}(\tfrac{1}{2}m\dot{\boldsymbol{x}}^2(u)+\tfrac{1}{2}m\Omega^2\boldsymbol{x}^2(u)+\boldsymbol{f}\,.\,\boldsymbol{x}(u))\,\mathrm{d}u\right). \tag{2.5}$$

Now (2.5) is the propagator of the Bloch equation for a forced harmonic oscillator and this can be found in the literature (see eg Papadopoulos 1969). With a constant external force $\boldsymbol{f}$ we have:

$$G_0(\boldsymbol{x},\beta|\boldsymbol{x}_0,0;\boldsymbol{f})$$
$$= \left(\frac{m\Omega}{2\pi\hbar\sinh(\beta\hbar\Omega)}\right)^{3/2}\exp\left[-\frac{m\Omega}{4\hbar}[\coth(\tfrac{1}{2}\beta\hbar\Omega)(\boldsymbol{x}-\boldsymbol{x}_0)^2+\tanh(\tfrac{1}{2}\beta\hbar\Omega)\right.$$
$$\left.\times(\boldsymbol{x}+\boldsymbol{x}_0)^2]-\frac{1}{\hbar\Omega}\tanh(\tfrac{1}{2}\beta\hbar\Omega)(\boldsymbol{x}+\boldsymbol{x}_0)\,.\,\boldsymbol{f}-\left(\frac{\tanh(\tfrac{1}{2}\beta\hbar\Omega)}{m\hbar\Omega^3}-\frac{\beta}{2m\Omega^2}\right)\boldsymbol{f}^2\right] \tag{2.6}$$

where in (2.6) we have written the term in the large square brackets in the exponential argument in a form convenient for the present evaluation.

Combining (2.4) and (2.6) and performing the integration over the auxiliary random force f we find:

$$G(\boldsymbol{x}, \beta|\boldsymbol{x}_0, 0) = \left(\frac{m\Omega}{2\pi\hbar \sinh(\beta\hbar\Omega)}\right)^{3/2} \left(\frac{\beta\hbar\Omega}{\tanh(\frac{1}{2}\beta\hbar\Omega)}\right)^{3/2} \exp\left(-\frac{m\Omega}{4\hbar}\coth(\tfrac{1}{2}\beta\hbar\Omega)(\boldsymbol{x}-\boldsymbol{x}_0)^2\right). \quad (2.7)$$

This is the required result; exact and in compact form.

As a final remark the present method provides the interpretation that the behaviour of a quantum memory oscillator equals the average behaviour of an assembly of externally forced quantum oscillators for which the external forces obey a gaussian distribution of zero mean and variance $2m\Omega^2kT$.

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